HOMEWORK 5

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Many of you didn't get the asymptotic behavior of $f(x) = \frac{x}{x^2+1}$ right. Obviously $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$, so the x-axis is an asymptote. In your sketches f often looked like a cubic function.

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Consider $r(\theta) = \theta + \sin^2\frac{\theta}{3} - 8$. This function is continuous and differentiable. We find $r'(\theta) = 1 - \frac{2}{3}\sin\frac{\theta}{3}\cos\frac{\theta}{3}$. Since $|\sin\frac{\theta}{3}\cos\frac{\theta}{3}| \le 1$ (in fact $\le \frac{1}{2}$, as can be shown using the addition formulas), we have $r'(\theta) \ge \frac{1}{3} > 0$.

Now we show:

(1) $r(\theta)$ has at least one zero. Since r is continuous, all we have to show is that r changes sign. But this is clear since $r(0) < 0$ and $r(10) > 0$.

(2) $r(\theta)$ has at most one zero. Assume not: then r has two zeros, say at $\theta = a$ and $\theta = b > a$. By the mean value theorem, there is a $c \in (a, b)$ such that $r'(c) = \frac{r(b)-r(a)}{b-a} = 0$. But this contradicts the fact that $r'(\theta) > 0$ everywhere.

The Ladder Problem.

The problem is to find the maximal ladder that can be carried round a corner of a corridor whose widths are $p = 8$ and $q = 6$.

1. Let's start with the first solution. At the moment the ladder touches the point with coordinates $(x, y) = (p, q)$, the ladder has length $L = a + b$, where $a = \frac{p}{\cos\theta}$ and $b = \frac{q}{\sin\theta}$, and where θ is the angle formed by the ladder and the x-axis.

Thus $L(\theta) = \frac{p}{\cos\theta} + \frac{q}{\sin\theta}$, hence $L'(\theta) = \frac{p\sin\theta}{\cos^2\theta} + \frac{-q\cos\theta}{\sin^2\theta} = 0$ if and only if $p\sin^3\theta_0 = q\cos^3\theta_0$, or $\tan\theta_0 = \sqrt[3]{q/p}$. This gives

$$\cos\theta_0 = \frac{\sqrt[3]{p}}{\sqrt{p^{2/3} + q^{2/3}}}, \quad \sin\theta_0 = \frac{\sqrt[3]{q}}{\sqrt{p^{2/3} + q^{2/3}}}.$$

Thus we get

$$L = (\sqrt{p^{2/3} + q^{2/3}}\,)(p^{2/3} + q^{2/3}) = (p^{2/3} + q^{2/3})^{3/2}.$$

This gives the maximal ladder. Or does it? Atually we find $L''(\theta_0) > 0$, hence L is a minimum! And it should be, as $L(\theta)$ becomes infinite as $\theta \to 0$ or $\theta \to \frac{\pi}{2}$. So something is wrong here.

2. The second solution offered goes like this: when the ladder touches the point (p, q), we have $\frac{a}{a+b} = \frac{p}{y}$ and $\frac{b}{a+b} = \frac{q}{x}$, hence $\frac{p}{y} + \frac{q}{x} = 1$, which finally gives $y = \frac{px}{x-q}$. The length ℓ of the ladder becomes maximal if and only if its square $\ell^2 = L = x^2 + y^2 = x^2 + \frac{p^2x^2}{(x-q)^2}$ does. Now $L'(x) = 0$ if and only if $x = 0$ (which is not meaningful physically) or $(x-q)^3 = p^2q$, i.e., $x_0 = \sqrt[3]{p^2q} + q$ and $y_0 = \sqrt[3]{pq^2} + p$, hence

$$\begin{aligned}
\ell^2 &= x_0^2 + y_0^2 \\
&= (p^2q)^{2/3} + 2q(p^2q)^{1/3} + q^2 + (pq^2)^{2/3} + 2p(pq^2)^{1/3} + p^2 \\
&= p^2 + 3p^{4/3}q^{2/3} + 3p^{2/3}q^{4/3} + q^2 \\
&= (p^{2/3} + q^{2/3})^3.
\end{aligned}$$

Note that this agrees with the above result. But since $\lim_{x\to q^+} L'(x) = -\infty$ and $\lim_{x\to\infty} L'(x) = +\infty$, the derivative changes sign at x_0 from $-$ to $+$, hence $L(x_0)$ is a local minimum. Again!

3. See
`http://archives.math.utk.edu/visual.calculus/3/applications.2/`
for the correct solution.

4. Observe that the result is the same as those above. And in fact, the solutions above can be turned into correct solutions with a little bit of additional reasoning.

Assume you have a ladder of variable length that always is as long as possible. As you move it around the corner, there will be one position in which the length is minimal, and this minimal length of the contractible ladder is the maximal length of a rigid ladder that can be moved around the corner.